

# Offset Circle Probabilities

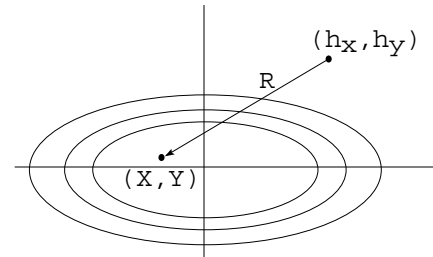
Ref: S&D p.1-8 to 1-10

## I. Problem Statement

A target's x- and y-positions are  $X \sim N(0, \sigma_X^2)$  and  $Y \sim N(0, \sigma_Y^2)$ .

$$R = \sqrt{(X-h_X)^2 + (Y-h_Y)^2}$$

= distance from an arbitrary point  $(h_X, h_Y)$  to  $(X, Y)$



What are  $F_R(r|h_X, h_Y)$  and  $f_R(r|h_X, h_Y)$ , the cumulative distribution and density functions for random variable  $R$ ?

## II. Development

Case 1:  $\sigma_X = \sigma_Y = \sigma$  and  $h = \sqrt{h_X^2 + h_Y^2} = 0$

$$\begin{aligned} F_R(z) &= P\{R \leq z\} \\ &= \left(1/2\pi\sigma^2\right) \iint_{x^2+y^2 \leq z^2} \exp(-(x^2+y^2)/2\sigma^2) dx dy \\ &= \left(1/2\pi\sigma^2\right) \int_0^{2\pi} \int_0^z \exp(-r^2/2\sigma^2) r dr d\theta \end{aligned}$$

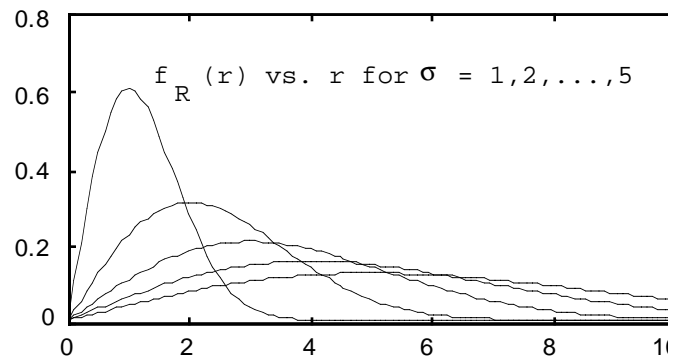
Integrating over  $\theta$  and making the change of variable  $u = r^2/2\sigma^2$ , so  $du = r dr/\sigma^2$  and  $dr = \sigma^2 du/r$ , we have

$$\begin{aligned} F_R(z) &= 1/\sigma^2 \int_{u=0}^{z^2/2\sigma^2} \exp(-u) \sigma^2 du \\ &= 1 - \exp(-z^2/2\sigma^2) \end{aligned}$$

So,

$$\begin{aligned} f_R(r) &= \frac{d}{dr} F_R(r) = \frac{r}{\sigma^2} \exp(-r^2/2\sigma^2) \quad (0) \\ &= \text{raylpdf}(r, \sigma) \end{aligned}$$

in Matlab w/ the Statistics Toolbox.



This is the Rayleigh density function.  $f_R(r)$  has a maximum value at  $r = \sigma$  and  $E[R] = \sigma\sqrt{\pi/2} \approx 1.2533 \sigma$ . It is the density function for the miss distance of a weapon with circular normal dispersion errors.

Applications:

1. A target's position is reported as a  $2\sigma$  circle. What is the probability that a target is located inside this circle?

$$F_R(2\sigma) = 1 - \exp(-(2\sigma)^2 / 2\sigma^2) \\ = 1 - \exp(-2) = .86$$

It's not difficult to show that the containment probability of any  $2\sigma$  ellipse is also .86.

2. In a dart game, let  $(X,Y)$  be where a dart hits.

$$P\{\text{bull's eye}\} = \{R \leq b\}$$

$$= F_R(b)$$

$$= 1 - \exp(-b^2 / 2\sigma^2)$$

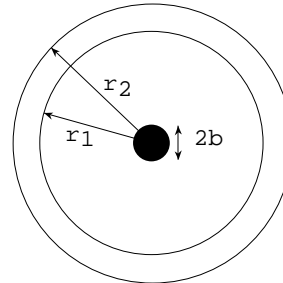
$$P\{\text{hit ring between } r_1 \text{ and } r_2\}$$

$$= P\{r_1 \leq R \leq r_2\}$$

$$= F_R(r_2) - F_R(r_1)$$

$$P\{\text{at least 1 bull's eye in 3 tosses}\}$$

$$= 1 - (1 - F_R(b))^3$$



Case 2: Given  $\sigma_x = \sigma_y = \sigma$  and

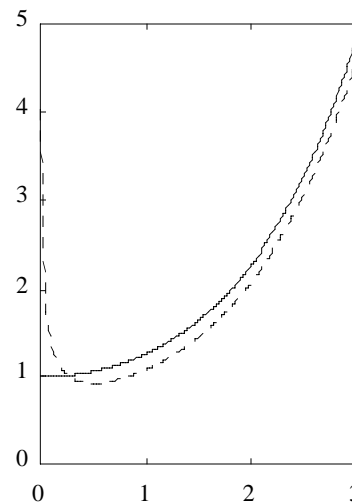
$$h = \sqrt{h_x^2 + h_y^2} > 0,$$

$$f_R(r) = \left(\frac{r}{\sigma^2}\right) \exp\left(\frac{-(r^2 + h^2)}{2\sigma^2}\right) I_0\left(\frac{rh}{\sigma^2}\right), \quad (1)$$

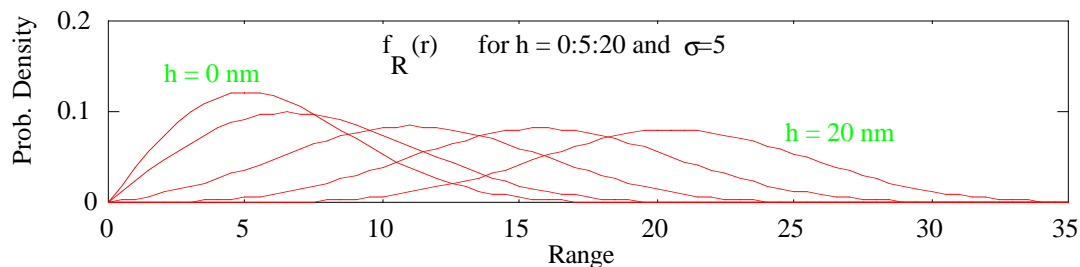
where  $I_0(x)$  is the modified Bessel function of order 0. Numerical integration is required to evaluate

$F_R(x)$ . For large  $x$ ,  $I_0(x) \approx e^x / \sqrt{2\pi x}$

In MATLAB,  $I_0(x) = \text{besseli}(0, x)$ .



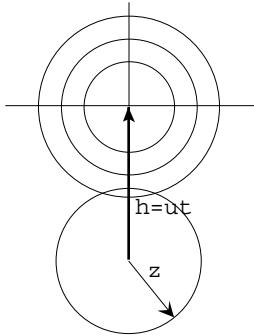
$I_0(x)$  (solid) and  $e^x/\sqrt{2\pi x}$  (dashed)



Applications:

### 1. Fleeing Normal Target

A target's distribution is circular BVN with standard deviation  $\sigma$ . At time 0, the target takes course  $\theta$  and speed  $u$ . At time  $t$ , a detection device with range  $z$  is placed at the origin. What is  $P_d$ ? If  $\theta=000^\circ$ , then by time  $t$  the target distribution is translated north a distance of  $h=ut$ . At that time,



$$P_d = \int_{r=0}^z f_R(r) dr \quad (2)$$

where  $f_R(r)$  is (1) and  $h=ut$ .

Note that since  $\sigma_x=\sigma_y=\sigma$ , the total probability mass covered by the sensor's detection disk is not a function of  $\theta$ . So (2) holds for any course  $\theta$  the target can select or any distribution of courses.

We can convert  $f_R(r)$  into a bivariate density in  $(R, \theta)$  as follows:

$$f_{R,\theta}(r, \theta) = f_R(r) / 2\pi r.$$

This is reasonable since the density at range  $r$  ( $f_R(r|h)$ ) is spread evenly around the circumference of a circle with radius  $r$  to give the bivariate density. This density looks like an expanding, circular ripple as  $h$  increases. Both  $f_R(r|h)$  and  $f_{R,\theta}(r, \theta|h)$  have maximum values near  $r=h$  for  $h \geq 2\sigma$ .

### 2. Radially Fleeing Normal Target

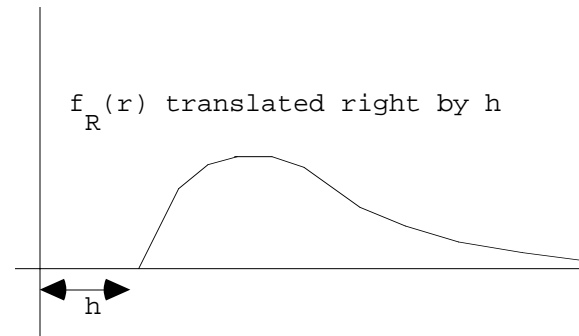
We have assumed so far that the target's selection of course  $\theta$  is independent of his  $(x, y)$  position. But if the target knows where datum (i.e. the origin) is and always flees radially away from that point, then the previous model is not appropriate. Instead, the density function for range from the datum starts as a Raleigh distribution (Eq.(0)) and simply translates to the right by  $h$  as time increases. So,

$$R \sim g(r) = \begin{cases} \left( \frac{r-h}{\sigma^2} \right) \exp\left( -\frac{(r-h)^2}{2\sigma^2} \right), & r > h \\ 0, & r \leq h \end{cases}$$

= raylpdf( $r-h, \sigma$ ) in Matlab

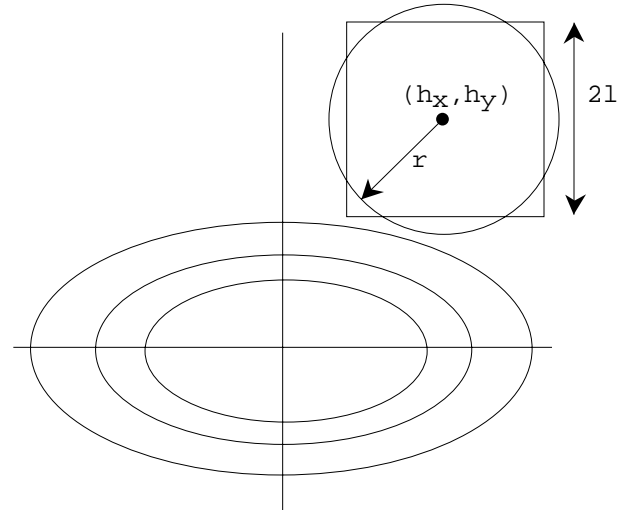
$$(R, \theta) \sim \frac{g(r)}{2\pi r}$$

With this model, there is a disk of radius  $h=ut$ , centered at the origin, having a target density of 0.



Case 3:  $\sigma_x \neq \sigma_y$  and  $h = \sqrt{h_x^2 + h_y^2} \geq 0$

As before, place the origin at the center of the translated target density; and let  $F_R(r|h_x, h_y)$  be the probability mass covered by a disk of radius  $r$  centered at  $(h_x, h_y)$ . We have no closed form expression for  $F_R$ , but can estimate its value as the probability mass covered by a square centered on  $(h_x, h_y)$  and having the same area as the disk. So,



$$F_R(r|h_x, h_y) \approx \int_{x=h_x-l}^{h_x+l} f_X(x) dx \int_{y=h_y-l}^{h_y+l} f_Y(y) dy$$

$$= \left[ \Phi\left(\frac{h_x+l}{\sigma_x}\right) - \Phi\left(\frac{h_x-l}{\sigma_x}\right) \right] \left[ \Phi\left(\frac{h_y+l}{\sigma_y}\right) - \Phi\left(\frac{h_y-l}{\sigma_y}\right) \right]$$

where  $(2l)^2 = \pi r^2$  or  $l = r\sqrt{\pi} / 2$ .

Applications:

1. A target has a BVN distribution centered at  $(0,0)$  with  $\sigma_x=10$  and  $\sigma_y=3$ . A sonobuoy with detection range 4 is placed at point  $(3,2)$ . So  $l = 4\sqrt{\pi} / 2 = 3.5449$ , and the approximate Pd is:

$$\begin{aligned} & (\Phi((3 + 1)/10) - \Phi((3 - 1)/10)) * (\Phi((2 + 1)/3) - \Phi((2 - 1)/3)) \\ &= (\Phi(.6545) - \Phi(-0.0545)) * (\Phi(1.8483) - \Phi(-0.5150)) \\ &= .1763 \end{aligned}$$

The correct value to 4 places is .1775.

## 2. Analytic Approximation for $\Phi(x)$

Let  $h_X=h_Y=0$  and  $\sigma_X=\sigma_Y=1$ . Then,

$$F_R(r|0,0) \approx (\Phi(1) - \Phi(-1))^2.$$

But from Case 1,

$$F_R(r|0,0) = 1 - \exp[-r^2/2].$$

So,

$$\begin{aligned} 1 - \exp[-r^2/2] &\approx (\Phi(1) - \Phi(-1))^2 \\ &= (\Phi(1) - (1 - \Phi(1)))^2 \\ &= (2\Phi(1) - 1)^2 \\ &= (2\Phi(\sqrt{\pi}r/2) - 1)^2 \end{aligned}$$

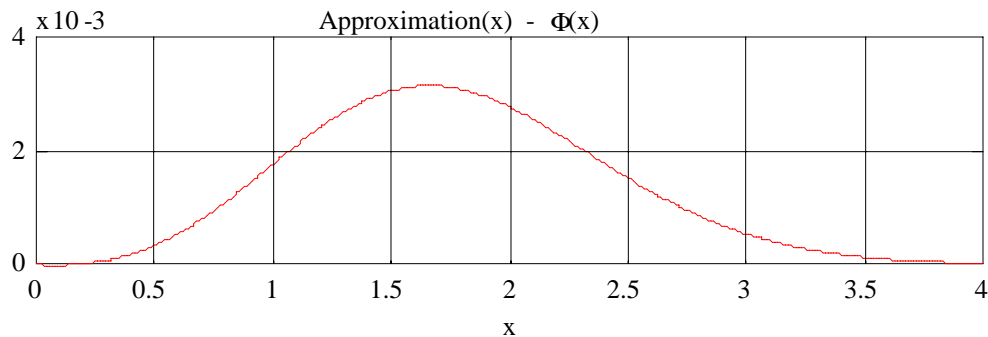
Or,

$$\Phi(\sqrt{\pi}r/2) \approx (1/2) (\sqrt{1 - \exp[-r^2/2]} + 1).$$

Now letting  $x = \sqrt{\pi}r/2$ , we have,

$$\Phi(x) \approx (1/2) \left( 1 + \sqrt{1 - \exp(-2x^2/\pi)} \right), x \geq 0$$

The approximation is smaller than  $\Phi(x)$  for  $0 \leq x \leq 1.5$  and is larger for  $x \geq 1.5$ , as indicated below. The maximum deviation from  $\Phi(x)$  is approximately .003146 and occurs at  $x=1.66$ . The approximation is always within .34% of  $\Phi(x)$ . That is,  $(|\text{Approx.} - \Phi|)/\Phi < .0034$ .



Revised: 7/24/98